ECE 3065 Homework 8: S-parameters

Solutions

1. The equivalent load as seen at port 1 is 100Ω, leading to a reflection coefficient of $s_{11} = +\frac{1}{3}$. The equivalent load as seen at port 2 is 25Ω, leading to a reflection coefficient of $s_{11} = -\frac{1}{3}$. To get s_{21}, start with the wave solution on the transmission line:

$$\tilde{v}(z) = K\left(\exp(-j\beta z) + \gamma \exp(+j\beta z)\right)$$

where $\gamma = -\frac{1}{3}$. On the left-hand side of the line, we evaluate this expression at $z = -\frac{\lambda}{4}$ and enforce the boundary condition:

$$\begin{aligned} V_1^+ + V_1^- &= \tilde{v}(z)\big|_{z=-\frac{\lambda}{4}} \\ (1+s_{11})V_1^+ &= K(-j + (-\frac{1}{3})j) \\ \frac{4}{3}V_1^+ &= -j\frac{4}{3}K \end{aligned}$$

Thus, $K = jV_1^+$. Evaluating at the end of the line, $\tilde{v}(0)$, we get:

$$\tilde{v}(0) = K(1 - \frac{1}{3}) = j\frac{2}{3}V_1^+$$

which is the voltage exiting port 2, V_2^-. We then know that $s_{21} = j\frac{2}{3}$.

Now let's try to find s_{12}. This is not a symmetrical circuit, so we cannot simply equate the result for s_{21}. Instead we note that the transmission line is essentially matched to a 50Ω load at its end when viewed from port 2. Thus, there will *only* be a forward-propagating voltage on the line and this is equivalent to the voltage V_1^- that exits port 1. The boundary condition, therefore, is much easier to enforce:

$$\text{Amplitude of wave into T-Line at Port 2} = (1+s_{22})V_2^+ = \frac{2}{3}V_2^+$$

We are not quite finished as this is not really s_{12} until we account for the quarter-wavelength phase change as the signal propagates down the line. This introduces a $-90°$ phase change, so that the actual value of $s_{12} = -j\frac{2}{3}$. The final S-matrix, in summary, is written below:

$$S = \begin{bmatrix} \frac{1}{3} & -j\frac{2}{3} \\ j\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

2. (a) By definition, a port's s-parameter is measured with a 50Ω "dummy" load connected to all of the other ports. The two port matrix of the new device with port 3 connected to a 50Ω load is just the truncation of the original S-matrix:

$$S' = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

(b) Start with the definition of a 3-port S-matrix:

$$\begin{bmatrix} V_1^- \\ V_2^- \\ V_3^- \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \end{bmatrix}$$

If port 3 is open, then the reflected coefficient back into the device is +1, implying that $V_3^+ = V_3^-$. From the S-matrix, we can write

$$V_3^+ = V_3^- = s_{31}V_1^+ + s_{32}V_2^+ + s_{33}V_3^+$$

Solving for V_3^+, we find that

$$V_3^+ = -\frac{1}{1-s_{33}}\left[s_{31}V_1^+ + s_{32}V_2^+\right]$$

Now let us rewrite the top two columns of the S-matrix:

$$\begin{aligned}
\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} &= \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \end{bmatrix} \\
&= \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} + \begin{bmatrix} s_{13} \\ s_{23} \end{bmatrix} V_3^+ \\
&= \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} - \frac{1}{1-s_{33}} \begin{bmatrix} s_{13} \\ s_{23} \end{bmatrix} \begin{bmatrix} s_{31} & s_{32} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \\
&= \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} - \frac{1}{1-s_{33}} \begin{bmatrix} s_{13}s_{31} & s_{13}s_{32} \\ s_{23}s_{31} & s_{23}s_{32} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \\
&= \underbrace{\left(\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} - \frac{1}{1-s_{33}} \begin{bmatrix} s_{13}s_{31} & s_{13}s_{32} \\ s_{23}s_{31} & s_{23}s_{32} \end{bmatrix} \right)}_{S'} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}
\end{aligned}$$

(c) This one is the trickiest. Start with the last two rows of a 3-port S-matrix:

$$\begin{bmatrix} V_2^- \\ V_3^- \end{bmatrix} = \begin{bmatrix} s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \end{bmatrix}$$

When port 2 and 3 are connected together, the outgoing wave from port 2 becomes an input for port 3 and vice versa. We can write this as:

$$\begin{bmatrix} V_3^+ \\ V_2^+ \end{bmatrix} = \begin{bmatrix} s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ V_3^+ \end{bmatrix}$$

$$\begin{bmatrix} V_2^+ \\ V_3^+ \end{bmatrix} = \begin{bmatrix} s_{31} \\ s_{21} \end{bmatrix} V_1^+ + \begin{bmatrix} s_{32} & s_{33} \\ s_{22} & s_{23} \end{bmatrix} \begin{bmatrix} V_2^+ \\ V_3^+ \end{bmatrix}$$

Solving for a single pair of V_2^+ and V_3^+ gives

$$\begin{bmatrix} V_2^+ \\ V_3^+ \end{bmatrix} = \left(\mathbf{I} - \begin{bmatrix} s_{32} & s_{33} \\ s_{22} & s_{23} \end{bmatrix} \right)^{-1} \begin{bmatrix} s_{31} \\ s_{21} \end{bmatrix} V_1^+$$

where **I** is a 2×2 identity matrix. Returning to the top parameter of the original S-matrix:

$$\begin{aligned} V_1^- &= s_{11}V_1^+ + \left[\, s_{12} \;\; s_{13} \,\right] \begin{bmatrix} V_2^+ \\ V_3^+ \end{bmatrix} \\ &= \underbrace{\left(s_{11} + \left[\, s_{12} \;\; s_{13} \,\right] \left(\mathbf{I} - \begin{bmatrix} s_{32} & s_{33} \\ s_{22} & s_{23} \end{bmatrix} \right)^{-1} \begin{bmatrix} s_{31} \\ s_{21} \end{bmatrix} \right)}_{s'_{11}} V_1^+ \end{aligned}$$

Of course, this s'_{11} is all we need to characterize a single-port device (this is a "1×1" scattering matrix). For anyone with the intestinal fortitude to work through the algebra without having a psychotic episode, this is the answer you would have found:

$$s'_{11} = s_{11} + \frac{s_{12}s_{31} - s_{12}s_{31}s_{23} - s_{12}s_{33}s_{21} + s_{13}s_{21} - s_{13}s_{21}s_{32} - s_{13}s_{22}s_{31}}{1 - s_{32} - s_{23} + s_{32}s_{23}}$$

Although these sorts of games can get very tricky and algebraic, they illustrate an important point: characterizing the s-parameters allows an engineer to predict the behavior of an altered-state device without having to re-measure the device or understand its inner-workings.