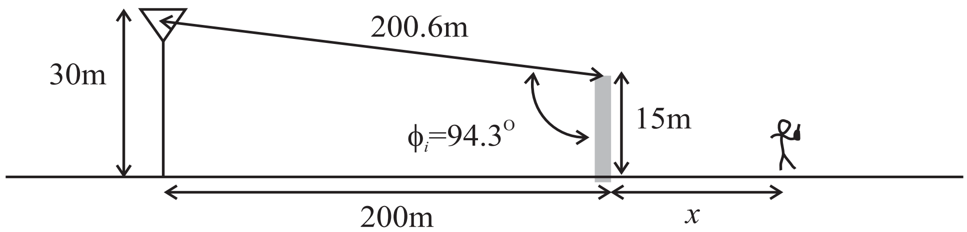
ECE 3065 Homework 6: Diffraction and Waveguides

Solutions

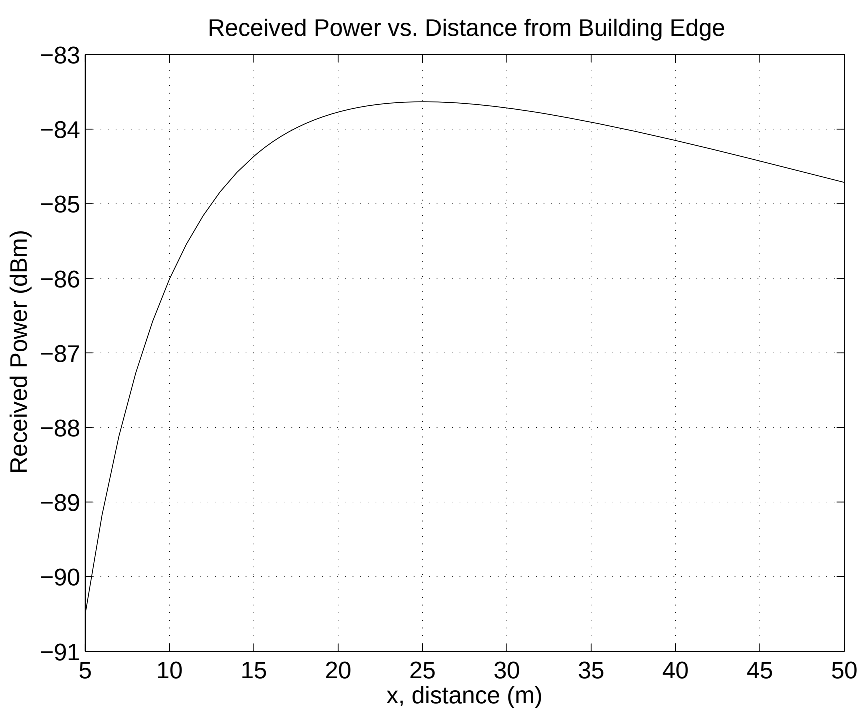
1. Below is a sketch of the screen equivalent problem:



The actual observation angle is given $\phi = \tan^{-1}(x/13.5)$. We note that the transmitter is vertically polarized, corresponding to $\|-$incidence in our problem. Thus, we may use the following diffraction coefficient:

$$D_{\|}(\phi, \phi_i) = \frac{-\exp\left(-j\frac{\pi}{4}\right)}{2\sqrt{2\pi}}\left[\sec\left(\frac{\phi - \phi_i}{2}\right) + \sec\left(\frac{\phi + \phi_i}{2}\right)\right]$$

The resulting graph is



Note that the signal performs better as the user steps away from the diffracting edge. However, there is a local maximum where the increased distance begins to introduce more loss simply due to extra distance traveled. This is a more graceful degradation, but certainly observable.

2. The general solution for the TM_n in the parallel plate waveguide is given by:

$$
\begin{aligned}
\tilde{E}_z(x,y,z) &= A_n \sin\left(\frac{n\pi y}{b}\right)\exp(-j\beta z) \\
\tilde{E}_y(x,y,z) &= \frac{-j\beta A_n}{h}\cos\left(\frac{n\pi y}{b}\right)\exp(-j\beta z) \\
\tilde{H}_x(x,y,z) &= \frac{j\omega\epsilon A_n}{h}\cos\left(\frac{n\pi y}{b}\right)\exp(-j\beta z)
\end{aligned}
$$

Now recall the boundary conditions for a perfect electric conductor:

$$\hat{\mathrm{n}} \cdot \tilde{\vec{\mathrm{E}}} = \tilde{\rho}_S \qquad \hat{\mathrm{n}} \times \tilde{\vec{\mathrm{H}}} = \tilde{\vec{\mathrm{J}}}_s$$

For the bottom plate, we evaluate these boundary conditions for $y = 0$ and $\hat{\mathrm{n}} = \hat{\mathrm{y}}$ to produce

$$\tilde{\rho}(z) = \frac{-j\beta A_n}{h}\exp(-j\beta z) \qquad \tilde{\vec{\mathrm{J}}}_s(z) = \frac{j\omega\epsilon A_n}{h}\exp(-j\beta z)\,\hat{\mathrm{z}}$$

For the top plate, we evaluate these boundary conditions for $y = b$ and $\hat{\mathrm{n}} = -\hat{\mathrm{y}}$ to produce

$$\tilde{\rho}(z) = \frac{j(-1)^n\beta A_n}{h}\exp(-j\beta z) \qquad \tilde{\vec{\mathrm{J}}}_s(z) = \frac{j(-1)^{n+1}\omega\epsilon A_n}{h}\exp(-j\beta z)\,\hat{\mathrm{z}}$$

Note that currents for TM modes flow forwards and backwards along the waveguide. Interestingly, for even modes the currents on the surface of the PEC flow equal and opposite to one another, just like a transmission line. Note: it is OK to omit the term $\exp(-j\beta z)$, even though it is incorrect and lazy – but of course your book did it in one of the solutions in the back.

3. The general solution for the TE_n in the parallel plate waveguide is given by:

$$
\begin{aligned}
\tilde{E}_x(x,y,z) &= \frac{j\omega\mu B_n}{h}\sin\left(\frac{n\pi y}{b}\right)\exp(-j\beta z) \\
\tilde{H}_y(x,y,z) &= \frac{j\beta B_n}{h}\sin\left(\frac{n\pi y}{b}\right)\exp(-j\beta z) \\
\tilde{H}_z(x,y,z) &= B_n \cos\left(\frac{n\pi y}{b}\right)\exp(-j\beta z)
\end{aligned}
$$

For the bottom plate, we evaluate boundary conditions for $y = 0$ and $\hat{\mathrm{n}} = \hat{\mathrm{y}}$ to produce

$$\tilde{\rho}(z) = 0 \qquad \tilde{\vec{\mathrm{J}}}_s(z) = B_n \exp(-j\beta z)\,\hat{\mathrm{x}}$$

For the top plate, we evaluate boundary conditions for $y = b$ and $\hat{\mathrm{n}} = -\hat{\mathrm{y}}$ to produce

$$\tilde{\rho}(z) = 0 \qquad \tilde{\vec{\mathrm{J}}}_s(z) = (-1)^{n+1} B_n \exp(-j\beta z)\,\hat{\mathrm{x}}$$

Note that currents for TM modes flow laterally in this waveguide. There is no surface charge. (The EM nerd will, at this point, verify that both solutions in (1) and (2) satisfy the continuity equation).

4. Start with the definition of the Poynting vector:

$$\begin{aligned}
\vec{\mathrm{S}}(x,y,z) &= \frac{1}{2}\mathrm{Real}\left\{\tilde{\vec{\mathrm{E}}}\times\tilde{\vec{\mathrm{H}}}\right\} \\
&= \frac{1}{2}\mathrm{Real}\left\{\left(\tilde{E}_z\,\hat{\mathrm{z}}+\tilde{E}_y\,\hat{\mathrm{y}}\right)\times\tilde{H}_x^*\,\hat{\mathrm{x}}\right\} \\
&= \frac{1}{2}\mathrm{Real}\left\{\left(\tilde{E}_z\,\hat{\mathrm{z}}+\tilde{E}_y\,\hat{\mathrm{y}}\right)\times\tilde{H}_x^*\,\hat{\mathrm{x}}\right\} \\
&= \frac{1}{2}\mathrm{Real}\left\{\tilde{E}_z\tilde{H}_x^*\,\hat{\mathrm{y}}-\tilde{E}_y\tilde{H}_x^*\,\hat{\mathrm{z}}\right\} \\
&= \frac{1}{2}\mathrm{Real}\left\{A_1\sin\left(\frac{\pi y}{b}\right)\times\frac{-j\omega\epsilon A_1}{h}\cos\left(\frac{\pi y}{b}\right)\hat{\mathrm{y}}-\frac{-j\beta A_1}{h}\cos\left(\frac{\pi y}{b}\right)\times\frac{-j\omega\epsilon A_1}{h}\cos\left(\frac{\pi y}{b}\right)\hat{\mathrm{z}}\right\} \\
&= \frac{1}{2}\mathrm{Real}\left\{\frac{-j\omega\epsilon A_1^2}{2h}\sin\left(\frac{2\pi y}{b}\right)\hat{\mathrm{y}}+\frac{\beta\omega\epsilon A_1^2}{h^2}\cos^2\left(\frac{\pi y}{b}\right)\hat{\mathrm{z}}\right\} \\
&= \frac{\beta\omega\epsilon A_1^2}{2h^2}\cos^2\left(\frac{\pi y}{b}\right)\hat{\mathrm{z}}
\end{aligned}$$

The power density flowing in the waveguide is a "bulge" in the center of the plates that propagates in the z-direction. Note that there were imaginary components that appeared in the Real{} operator that vanished in the final Poynting vector; these components represent stored energy that does not have a net direction of propagation.

5. Here is the basic procedure for finding the dimensions. First, we know that TE_{10} is the dominant mode. Using our cut-off equation:

$$(f_c)_{10} = \frac{1}{2\sqrt{\mu_0\epsilon_0}}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{0}{b}\right)^2} = \frac{1.5\times10^8\ \mathrm{m/s}}{a} = 3\ \mathrm{GHz}$$

From this we know that a is 5cm.

Note that the next cut-off frequency of 6 GHz is *twice* the dominant mode. This must correspond to TE_{20} mode:

$$(f_c)_{20} = \frac{1}{2\sqrt{\mu_0\epsilon_0}}\sqrt{\left(\frac{2}{0.05\ \mathrm{m}}\right)^2+\left(\frac{0}{b}\right)^2} = 6\ \mathrm{GHz}$$

The next-highest cut-off frequency of 7.5 GHz is clearly *not* three times the dominant mode. This must correspond to the TE_{01} mode:

$$(f_c)_{01} = \frac{1}{2\sqrt{\mu_0\epsilon_0}}\sqrt{\left(\frac{0}{a}\right)^2+\left(\frac{1}{b}\right)^2} = \frac{1.5\times10^8\ \mathrm{m/s}}{b} = 7.5\ \mathrm{GHz}$$

Which results in a b of 2cm.

At this point, the clever student will want to double check that the TE_{01} and TE_{20} modes are not degenerate. This would occur for $b = 2.5$cm and would mean that the 7.5 GHz cut-off frequency would actually correspond to the $\mathrm{TE/TM}_{11}$ modes. For $a = 5$cm and $b = 2.5$cm, we calculate

$$(f_c)_{11?} = \frac{1}{2\sqrt{\mu_0\epsilon_0}}\sqrt{\left(\frac{1}{0.05}\right)^2+\left(\frac{1}{0.025}\right)^2} = 6.7\ \mathrm{GHz}$$

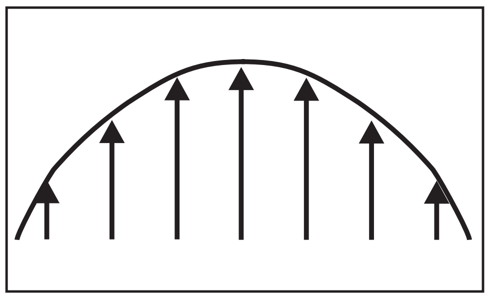
Since this is not listed as one of the first three cut-off frequencies, we can be assured that our answer is correct. The actual cut-off for the TE/TM_{11} modes is

$$(f_c)_{11} = \frac{1}{2\sqrt{\mu_0 \epsilon_0}} \sqrt{\left(\frac{1}{0.05}\right)^2 + \left(\frac{1}{0.02}\right)^2} = 8.1 \text{ GHz}$$

which would have been the fourth cut-off frequency.

The field distribution for the TE_{10} is sketched below. The fields are uniform from top to bottom with vertical polarization. Their amplitudes taper sinusoidally across the lateral dimension of the waveguide.

Waveguide Cross-section



6. **Circular vs. Rectangular Waveguide:**

   (a) For a given cut-off frequency f, the perimeter of the circular waveguide is:

   $$r = \frac{0.293}{f\sqrt{\mu\epsilon}} \longrightarrow \text{Perimeter} = 2\pi r = \frac{1.84}{f\sqrt{\mu\epsilon}}$$

   For a square wave with the same cut-off frequency:

   $$a = \frac{0.5}{f\sqrt{\mu\epsilon}} \longrightarrow \text{Perimeter} = 4a = \frac{2.00}{f\sqrt{\mu\epsilon}}$$

   Clearly the circular guide is cheaper.

   (b) The ratio between the second and the first mode cut-offs for the circular guide is:

   $$\frac{\text{Cut-off for TM}_{01}}{\text{Cut-off for TE}_{11}} = 1.31$$

   The ratio between the second and the first mode cut-offs for the square waveguide is:

   $$\frac{\text{Cut-off for TE/M}_{11}}{\text{Cut-off for TE}_{10}} = 1.41$$

   For the same initial cut-off frequency, the square guide should have more single-mode bandwidth.