ECE 3065 Homework 4: Plane Waves

Solutions

1. **Plane Wave Solutions:**

(a)

$$\begin{aligned}\tilde{\vec{E}}(\vec{r}) &= E_0\hat{e}\exp\left(-jk\hat{k}\cdot\vec{r}\right) \\ &= E_0\hat{e}\exp\left(-jkk_xx - jkk_yy - jkk_zz\right)\end{aligned}$$

Now substitute this into the wave equation:

$$\begin{aligned}\left(\nabla^2 + k^2\right)\tilde{\vec{E}} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{\vec{E}} \\ &= E_0\hat{e}\left[-k^2k_x^2 - k^2k_y^2 - k^2k_z^2 + k^2\right]\exp\left(-jkk_xx - jkk_yy - jkk_zz\right) \\ &= E_0k^2\hat{e}\left[-\|\hat{k}\|^2 + 1\right]\exp\left(-jkk_xx - jkk_yy - jkk_zz\right) \\ &= 0\end{aligned}$$

(b) The full solution to this plane wave, according to the generic form we learned in class, is

$$\begin{aligned}\tilde{\vec{E}}(\vec{r}) &= \overbrace{10\sqrt{14}}^{E_0}\left[\overbrace{\frac{1}{\sqrt{6}}\hat{x} + \frac{2}{\sqrt{6}}\hat{y} - \frac{1}{\sqrt{6}}\hat{z}}^{\hat{e}}\right]\exp\left(-j\overbrace{332}^{k}\left[\frac{3}{\sqrt{11}}\hat{x} - \frac{1}{\sqrt{11}}\hat{y} + \frac{1}{\sqrt{11}}\hat{z}\right]\cdot\vec{r}\right)\ \text{mV/m} \\ \tilde{\vec{H}}(\vec{r}) &= \underbrace{65.0}_{H_0}\underbrace{\left[-\frac{1}{\sqrt{66}}\hat{x} + \frac{4}{\sqrt{66}}\hat{y} + \frac{7}{\sqrt{66}}\hat{z}\right]}_{\hat{h}}\exp\left(-j332\underbrace{\left[\frac{3}{\sqrt{11}}\hat{x} - \frac{1}{\sqrt{11}}\hat{y} + \frac{1}{\sqrt{11}}\hat{z}\right]}_{\hat{k}}\cdot\vec{r}\right)\ \mu\text{A/m}\end{aligned}$$

Since $k = 332$ rad/m, the wavelength of radiation must be $\lambda = \frac{2\pi}{k} = 0.0189$m. The angle-of-arrival coordinates come from the following equations:

$$\begin{aligned}\theta &= \cos^{-1} -k_z \\ \varphi &= \tan^{-1}\frac{k_y}{k_x} \qquad \text{(add } \pi \text{ if } k_x < 0\text{)}\end{aligned}$$

For this plane wave, the angle-of-arrival coordinates are $\theta = 107.5°$ (elevation, measured from z-axis) and $\varphi = 18.4°$ (azimuth).

2. **Reflection:** Here is an outline for the solution to parallel polarization. Follow Table 1 at the end of the solution set for reference.

Define Incident Fields: Using the generic form of a plane wave, we express both incident electric and magnetic fields as:

$$\begin{aligned} \tilde{\vec{\mathsf{E}}}_i(\vec{\mathrm{r}}) &= E_i \overbrace{[\cos\theta_i \hat{\mathrm{x}} - \sin\theta_i \hat{\mathrm{z}}]}^{\hat{\mathrm{e}}_i} \exp\big(-jk_1 \overbrace{[\sin\theta_i \hat{\mathrm{x}} + \cos\theta_i \hat{\mathrm{z}}]}^{\hat{\mathrm{k}}_i} \cdot \vec{\mathrm{r}}\big) \\ \tilde{\vec{\mathsf{H}}}_i(\vec{\mathrm{r}}) &= \frac{E_i}{\eta_1} \underbrace{\hat{\mathrm{y}}}_{\hat{\mathrm{h}}_i} \exp\big(-jk_1 \underbrace{[\sin\theta_i \hat{\mathrm{x}} + \cos\theta_i \hat{\mathrm{z}}]}_{\hat{\mathrm{k}}_i} \cdot \vec{\mathrm{r}}\big) \end{aligned}$$

where $\vec{\mathrm{r}} = x\hat{\mathrm{x}} + y\hat{\mathrm{y}} + z\hat{\mathrm{z}}$.

Outline Solution Regions: We will define to solution regions – one for each medium – which will result in two separate solutions. The first solution will be valid for all points in space with $z < 0$, while the second solution will be valid for all $z > 0$.

Posit Solution Basis: We will posit a single plane wave to be the solution basis in each of the two solution regions. Let solution basis 1 be the reflected wave in medium 1, which takes the following mathematical form:

$$\begin{aligned} \tilde{\vec{\mathsf{E}}}_r(\vec{\mathrm{r}}) &= E_r \overbrace{[\cos\theta_r \hat{\mathrm{x}} + \sin\theta_r \hat{\mathrm{z}}]}^{\hat{\mathrm{e}}_r} \exp\big(-jk_1 \overbrace{[\sin\theta_r \hat{\mathrm{x}} - \cos\theta_r \hat{\mathrm{z}}]}^{\hat{\mathrm{k}}_r} \cdot \vec{\mathrm{r}}\big) \\ \tilde{\vec{\mathsf{H}}}_r(\vec{\mathrm{r}}) &= \frac{E_r}{\eta_1} \underbrace{[-\hat{\mathrm{y}}]}_{\hat{\mathrm{h}}_r} \exp\big(-jk_1 \underbrace{[\sin\theta_r \hat{\mathrm{x}} - \cos\theta_r \hat{\mathrm{z}}]}_{\hat{\mathrm{k}}_r} \cdot \vec{\mathrm{r}}\big) \end{aligned}$$

Let solution basis 2 be the transmitted wave in medium 2, which takes this form:

$$\begin{aligned} \tilde{\vec{\mathsf{E}}}_t(\vec{\mathrm{r}}) &= E_t \overbrace{[\cos\theta_t \hat{\mathrm{x}} - \sin\theta_t \hat{\mathrm{z}}]}^{\hat{\mathrm{e}}_t} \exp\big(-jk_2 \overbrace{[\sin\theta_t \hat{\mathrm{x}} + \cos\theta_t \hat{\mathrm{z}}]}^{\hat{\mathrm{k}}_t} \cdot \vec{\mathrm{r}}\big) \\ \tilde{\vec{\mathsf{H}}}_t(\vec{\mathrm{r}}) &= \frac{E_t}{\eta_2} \underbrace{\hat{\mathrm{y}}}_{\hat{\mathrm{h}}_t} \exp\big(-jk_2 \underbrace{[\sin\theta_t \hat{\mathrm{x}} + \cos\theta_t \hat{\mathrm{z}}]}_{\hat{\mathrm{k}}_t} \cdot \vec{\mathrm{r}}\big) \end{aligned}$$

These equations are simply generic plane wave expressions; we have made no assumptions about their directions of propagation or their amplitudes.

Enforce Boundary Conditions: This step is where the rubber hits the road. We will enforce two different boundary conditions, starting with the tangential electric field.

Tangential Electric Field: At the interface, the tangent components of electric field must be continuous across the dielectric interface. We may write this condition as

$$\left(\tilde{\vec{\mathsf{E}}}_i(\vec{\mathrm{r}})\Big|_{z=0} + \tilde{\vec{\mathsf{E}}}_r(\vec{\mathrm{r}})\Big|_{z=0}\right)\cdot\hat{\mathrm{x}} = \left(\tilde{\vec{\mathsf{E}}}_i(\vec{\mathrm{r}})\Big|_{z=0}\right)\cdot\hat{\mathrm{x}}$$

Plugging in our incident, reflected, and transmitted wave expressions yields the following result:

$$E_i\cos\theta_i\exp(-jk_1\sin\theta_i\cdot x) + E_r\cos\theta_r\exp(-jk_1\sin\theta_r\cdot x) = E_t\cos\theta_t\exp(-jk_2\sin\theta_t\cdot x)$$

This condition must hold across the entire boundary, yet both sides of the equation are functions of position x. There are only two ways mathematically for this to be satisfied. Either all of the amplitudes are zero (untrue since a non-zero incident field is given in the problem statement) or all of the exponential arguments must be equal to the same value. Thus, we can write

$$k_1\sin\theta_i = k_1\sin\theta_r = k_2\sin\theta_t \quad\longrightarrow\quad \begin{array}{ll}\text{Reflection:} & \theta_i = \theta_r \\ \text{Refraction:} & \frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}\end{array}$$

We have just used Maxwell's equations and the dielectric boundary conditions to prove Snell's Laws of Reflection and Refraction! If we allow for these simplifications, the simplified expression of the boundary condition is given by

$$E_i\cos\theta_i + E_r\cos\theta_i = E_t\cos\theta_t$$

We will need to enforce at least one more boundary condition before the solution is achieved.

Tangential Magnetic Field: Now let us enforce the boundary condition for the tangential magnetic field components. The total tangential magnetic field must be equal on either side of the dielectric:

$$\left(\tilde{\vec{\mathsf{H}}}_i(\vec{\mathrm{r}})\Big|_{z=0} + \tilde{\vec{\mathsf{H}}}_r(\vec{\mathrm{r}})\Big|_{z=0}\right)\cdot\hat{\mathrm{y}} = \left(\tilde{\vec{\mathsf{H}}}_i(\vec{\mathrm{r}})\Big|_{z=0}\right)\cdot\hat{\mathrm{y}}$$

Making the same simplifications as the tangential electric field boundary condition, we ultimately arrive at

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

Now we have two equations and two unknowns, E_r and E_t. The incident amplitude E_i is given, the angles θ_i, θ_r, and θ_t are solvable by Snell's laws, and the other terms are material constants.

Simplify and Approximate: Scattered waves present elegant problems to the engineer that deserve elegant solutions. An important part of any solution in wave propagation is to order and simplify the result so that it is compact, communicates physical insight, and is easily applicable to similar problems.

$$\begin{array}{rcrcl}\Gamma_\parallel & - & \frac{\cos\theta_t}{\cos\theta_i}\tau_\parallel & = & -1 \\ \Gamma_\parallel & + & \frac{\eta_1}{\eta_2}\tau_\parallel & = & 1\end{array} \quad\longrightarrow\quad \begin{array}{l}\Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} \\ \tau_\parallel = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}\end{array} \tag{1}$$

3. **Stealth Problem:** This looks like a very complicated problem unless we view each propagation medium as a transmission line and view this entire problem as a quarter-wavelength splice match. Imagine that we have a transmission line that is very long with $Z_0 = 377\Omega$ (the air). We will place a splice some distance from the $215 + j197\Omega$ load (0.57+j0.52 when normalized)

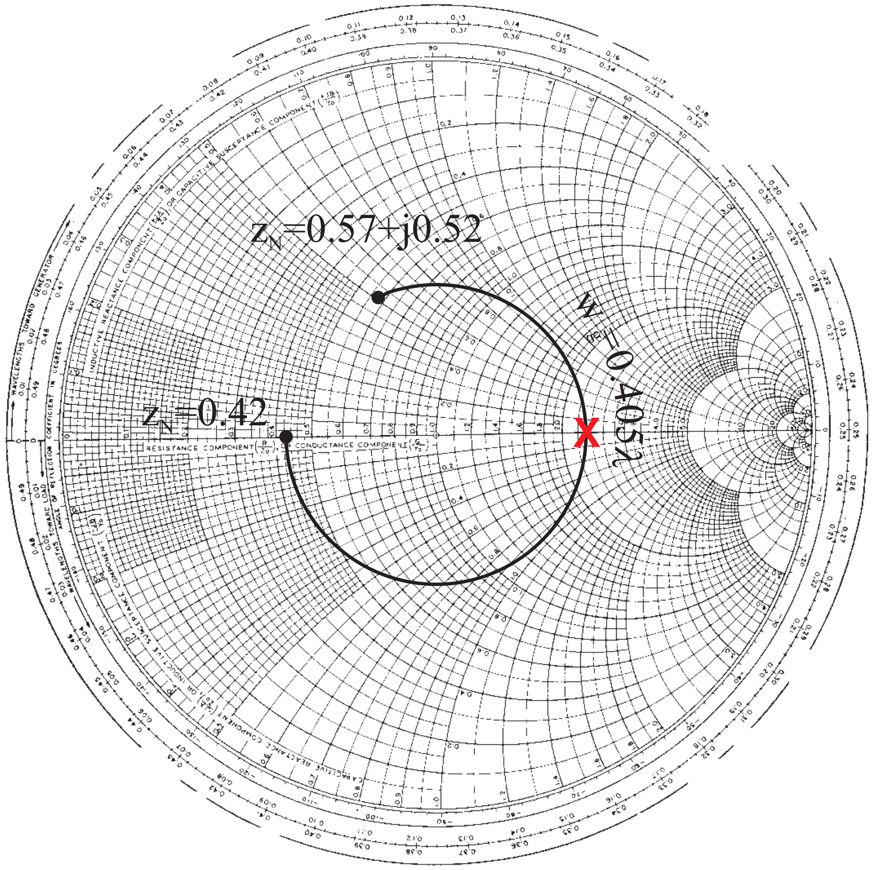


Figure 1: Smith chart calculation for the quarter-wavelength "stealthy" splice.

that transforms this complex load to a real-valued equivalent impedance. See Figure 1 for this calculation. It turns out that we require 0.405λ of the 377Ω transmission line (the air gap) to transform the "load" to a real-valued impedance. At 15 GHz, that means $w_g = 8.1$mm.

The transformed load has a normalized input impedance of 0.42 or, when unnormalized, is 158Ω. Thus, our quarter-wavelength splice should have an intrinsic impedance of $\eta_M = \sqrt{(158\Omega)(377\Omega)} = 244\Omega$. Since our medium is non-magnetic, we will assume μ_0 and calculate ϵ_r from the following relationship:

$$\eta_M = \frac{\mu_0}{\epsilon_r \epsilon_0} \longrightarrow \epsilon_r = 2.4$$

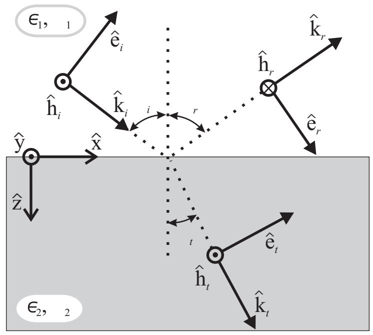
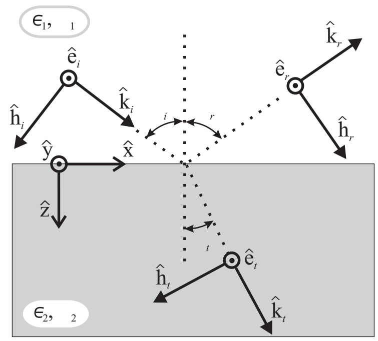
Now that we know the permittivity, we can calculate the dimension w_M:

$$w_M = \frac{1}{4}\lambda = \frac{1}{4}\frac{1}{f\sqrt{\mu_0 \epsilon_r \epsilon_0}} = 3.2\text{mm}$$

You'll notice that, unlike other splice problems, we did not use the closest intersection point on the Smith chart's real axis (the red X on Figure 1). Had we chosen this point, we would have found that the resulting ϵ_r was actually *less* than 1, which is physically impossible.

One final note: think how difficult the stealth problem is. One must be able to design a material that is non-reflective at *all possible* radar frequencies. A high-quality radar is capable of detecting the radar cross-section of a bumble bee. So a stealthy ship or aircraft must scatter less radio power than a bumble bee.

Table 1: Formula summary of Fresnel reflection from a dielectric slab.

$\parallel$ **Polarization**	$\perp$ **Polarization**
	
$\Gamma_\parallel = \dfrac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$	$\Gamma_\perp = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$
$\tau_\parallel = \dfrac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = \dfrac{\cos\theta_i}{\cos\theta_t}\left(1+\Gamma_\parallel\right)$	$\tau_\perp = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = 1+\Gamma_\perp$
$\hat{k}_i = \sin\theta_i \hat{x} + \cos\theta_i \hat{z}$ $\hat{e}_i = \cos\theta_i \hat{x} - \sin\theta_i \hat{z}$ $\hat{h}_i = \hat{y}$	$\hat{k}_i = \sin\theta_i \hat{x} + \cos\theta_i \hat{z}$ $\hat{e}_i = \hat{y}$ $\hat{h}_i = -\cos\theta_i \hat{x} + \sin\theta_i \hat{z}$
$\hat{k}_r = \sin\theta_r \hat{x} - \cos\theta_r \hat{z}$ $\hat{e}_r = \cos\theta_r \hat{x} + \sin\theta_r \hat{z}$ $\hat{h}_r = -\hat{y}$	$\hat{k}_r = \sin\theta_r \hat{x} - \cos\theta_r \hat{z}$ $\hat{e}_r = \hat{y}$ $\hat{h}_r = \cos\theta_r \hat{x} + \sin\theta_r \hat{z}$
$\hat{k}_t = \sin\theta_t \hat{x} + \cos\theta_t \hat{z}$ $\hat{e}_t = \cos\theta_t \hat{x} - \sin\theta_t \hat{z}$ $\hat{h}_t = \hat{y}$	$\hat{k}_t = \sin\theta_t \hat{x} + \cos\theta_t \hat{z}$ $\hat{e}_t = \hat{y}$ $\hat{h}_t = -\cos\theta_t \hat{x} + \sin\theta_t \hat{z}$

General Plane Wave Solution

$$\vec{E}_\diamond(\vec{r}) = E_\diamond\, \hat{e}_\diamond\, \exp\left(j\left[\phi - k\hat{k}_\diamond \cdot \vec{r}\right]\right) \qquad \vec{H}_\diamond(\vec{r}) = \frac{E_\diamond}{\eta}\hat{h}_\diamond \exp\left(j\left[\phi - k\hat{k}_\diamond \cdot \vec{r}\right]\right)$$

$\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$ $\quad \eta = \sqrt{\frac{\mu}{\epsilon}}$ $\quad k = \frac{2\pi}{\lambda} = \frac{2\pi f}{v_p}$ $\quad \diamond \rightarrow i$[ncident] **or** r[eflected] **or** t[ransmitted]

Snell's Law of Reflection	**Snell's Law of Refraction**
$\theta_i = \theta_r$	$\dfrac{\sin\theta_i}{v_{p1}} = \dfrac{\sin\theta_t}{v_{p2}}$ where $v_p = \dfrac{1}{\sqrt{\epsilon\mu}}$

Physical Quantities

θ_i	angle of incidence	E	electric field amplitude (V/m)
θ_r	angle of reflection	$\Gamma_{\parallel,\perp}$	reflection coefficient ($\frac{E_r}{E_i}$)
θ_t	angle of transmission	$\tau_{\parallel,\perp}$	transmission coefficient ($\frac{E_t}{E_i}$)
$\hat{e}$	electric field polarization	η	intrinsic impedance (Ω, Ohms)
$\hat{h}$	magnetic field polarization	v_p	velocity of propagation (m/s)
$\hat{k}$	direction of propagation	k	wavenumber (radians/m)
μ	magnetic permeability (H/m)	λ	wavelength (m)
ϵ	electric permittivity (F/m)	f	frequency (Hz)