

# A Basic Relationship Between Multipath Angular Spread and Narrowband Fading in a Wireless Channel <sup>\*</sup>

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## Abstract

This paper develops a novel relationship between azimuthal distribution of multipath power and narrowband small-scale fading characteristics. The fundamental relationship is useful for studying adaptive arrays, smart antennas, equalization, diversity, and any other wireless technology or concept that depends on the spatial characteristics of radio and microwave propagation.

## 1 Introduction

As mobile communication systems become more advanced, their design and optimization become increasingly affected by the spatial parameters of the wireless

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channel. This paper presents a novel definition of multipath angular spread, based on the distribution of incoming multipath power about the horizon. The paper then relates this definition to the small-scale fading experienced by a mobile receiver. The end result is a simple, intuitively appealing equation for studying fading in multipath channels. The result is also pragmatic, illustrating ways to imply angle-of-arrival multipath characteristics from incoherent, omnidirectional fading measurements *or* to imply fading behavior from angle-of-arrival measurements.

## 2 Definition of Angular Spread

For typical terrestrial propagation, radio waves arrive at the receiver from a number of azimuthal directions about the horizon [1]. This distribution of multipath power is conveniently described by the function,  $p(\theta)$ , where  $\theta$  is the azimuthal angle. In the limit of very small  $d\theta$ , the term  $p(\theta)d\theta$  represents the power of a single multipath plane wave received from the  $\theta$  direction by the receiver [1].

We propose a new method for quantifying the angular spread of multipath power which is based on the Fourier coefficients of  $p(\theta)$ . Let the value for angular spread,  $\Lambda$ , be defined by the following:

$$\Lambda = \sqrt{1 - \frac{|F_1|^2}{|F_0|^2}}, \quad F_n = \int_0^{2\pi} p(\theta) \exp(jn\theta) d\theta \quad (1)$$

where  $F_n$  is the  $n$ th complex Fourier coefficient of  $p(\theta)$ . There are several advantages to defining angular spread in this manner. First, since angular spread is normalized by  $|F_0|$  (the total amount of received power), it is invariant under changes in transmitted power. Second,  $\Lambda$  is invariant under any series of rotational or symmetrical transformations of  $p(\theta)$ . Finally, this definition is intuitive; angular spread ranges from 0 to 1, with 0 denoting the extreme case of a single multipath component from a single direction and 1 denoting no clear bias in the angular distribution of received power.

As just one example of how to use this definition, consider a situation where diffuse multipath power is arriving uniformly over a continuous range of azimuthal angles. The function  $p(\theta)$  for this channel description is

$$p(\theta) = \begin{cases} \frac{P_T}{\alpha} & : \theta_o \leq \theta \leq \theta_o + \alpha \\ 0 & : \text{elsewhere} \end{cases} \quad (2)$$

The angle  $\alpha$  indicates the width of the sector (in radians) of arriving multipath power and the angle  $\theta_o$  is any arbitrary offset, as illustrated by Figure 1. A total amount of power,  $P_T$ , arrives uniformly from all directions within the range of azimuthal angles; no multipath power arrives from outside this range. The angular spread from Eqn (1) is

$$\Lambda = \sqrt{1 - \frac{4}{\alpha^2} \sin^2\left(\frac{\alpha}{2}\right)} \quad (3)$$

The limiting cases of Eqn (3) provide deeper understanding of this definition of angular spread. The limiting case of a single multipath arriving from precisely one direction corresponds to  $\alpha = 0$ , which results in  $\Lambda = 0$ . The other limiting case of uniform illumination in all directions corresponds to  $\alpha = 2\pi$ , which results in the maximum angular spread of 1.

### 3 Fading Rate Variance

In a local area, propagation at microwave frequencies can be represented as the superposition of numerous plane waves arriving from the horizon [2, 3]. The resulting voltage at the terminal of a receiver can then be represented by

$$\tilde{V}(\vec{x}) = \sum_{i=1}^N V_i \exp\left(j\Phi_i - j\vec{k}_i \cdot \vec{x}\right) \quad (4)$$

where  $N$  is the total number of multipath components,  $V_i$  is the magnitude of the received voltage of the  $i$ th multipath wave,  $\vec{k}_i$  is the wavevector of the  $i$ th multipath wave,  $\vec{x}$  is a two-dimensional position vector, and  $\Phi_i$  is the phase of the  $i$ th multipath wave, which is usually taken to be a uniformly distributed random variable on the

interval  $[0, 2\pi)$ . This is a standard representation of mobile propagation in a local area where the amplitudes of arriving voltages,  $V_i$ , remain constant over position changes less than 10 wavelengths (log-normal shadowing is not an issue) [3, 4, 5]. The wavevector  $\vec{k}_i$  points in the direction of the  $i$ th multipath and always has a magnitude of  $k$ , which is related to the wavelength of radiation ( $k = \frac{2\pi}{\lambda}$ ).

To study the fading of a receiver *in motion*, Eqn (4) must be converted to a function of time by making the position vector a function of time,  $t$ , and mobile velocity magnitude and direction,  $\vec{v}$ . Substituting the basic equation of motion,  $\vec{x} = \vec{v}t$ , leads to

$$\tilde{V}(t) = \sum_{i=1}^N V_i \exp(j\Phi_i) \exp[-j t (\vec{k}_i \cdot \vec{v})] \quad (5)$$

The received narrowband fading signal *power*, in terms of Eqn (5), is given by

$$P(t) = |\tilde{V}(t)|^2 = \sum_{i=1}^N \sum_{j=1}^N V_i V_j \cos [(\Phi_i - \Phi_j) + t \vec{v} \cdot (\vec{k}_i - \vec{k}_j)] \quad (6)$$

Eqn (6) is *normalized* power with units of *volts-squared*. We define the variance of the fading rate to be

$$\sigma^2 = \text{E} \left\{ \left( \frac{dP(t)}{dt} \right)^2 \right\} - \left( \text{E} \left\{ \frac{dP(t)}{dt} \right\} \right)^2 \quad (7)$$

Upon inspection, the right term on the right-hand side of Eqn (7) is zero for the power representation in Eqn (6). The ensemble average of the remaining term is taken over the  $N$  random phases,  $\Phi_i$ . The fading rate variance becomes

$$\sigma^2 = \sum_{i=1}^N \sum_{j=1}^N P_i P_j \text{E} \left\{ \left[ \vec{v} \cdot (\vec{k}_i - \vec{k}_j) \right]^2 \right\} \quad (8)$$

where  $P_i = V_i^2$  and  $P_j = V_j^2$ ;  $P_i$  is the power of the  $i$ th individual multipath wave received in the absence of other multipath.

To evaluate Eqn (8) further, the velocity vector needs to be described in greater detail. For convenience, the velocity will be described as having a constant magnitude,  $v$ , and a random direction, uniformly distributed about the azimuth. The

physical interpretation of this scenario is appealing:  $\sigma^2$  represents the *mean-square* fading rate a receiver will experience in a local area, without presuming an exact direction of travel. Furthermore, this definition of  $\sigma^2$  is mathematically equivalent to the average rate of squared power change as measured along any two orthogonal tracks in the local area, assuming the same mobile receiver speed. Figure 2 illustrates this interpretation.

After evaluating the final ensemble average, the fading rate variance becomes

$$\sigma^2 = 2k^2v^2 \sum_{i=1}^N \sum_{j=1}^N P_i P_j \sin^2 \left( \frac{\theta_i - \theta_j}{2} \right) \quad (9)$$

where  $\theta_i$  and  $\theta_j$  are the arrival angles of the  $i$ th and  $j$ th multipath wave, respectively. An even more convenient form for Eqn (9) may be obtained by using the angular distribution of power for a set of propagating plane waves:

$$p(\theta) = \sum_{i=1}^N P_i \delta(\theta - \theta_i) \quad (10)$$

where  $\delta(\theta)$  is the unit impulse function. Eqn (9) may be recast in terms of angular spread,  $\Lambda$  (see Appendix for more detail):

$$\sigma = kv\Lambda P_T \quad (11)$$

where  $P_T$  is the average local area power seen by the mobile. Although the relationship was derived from discrete multipath, Eqn (11) is equally valid for continuous, diffuse power distributions.

Eqn (11) is a remarkable result; it states that as multipath power becomes more spread out about the horizon, the average fading rate in a local area will increase *exactly proportional* to the angular spread. This type of fading behavior is expected in the literature [2], although a simple relationship such as Eqn (11) has not previously existed to describe the phenomenon.

## 4 Conclusions

This paper has presented a fundamental relationship between the angular distribution of power in a multipath channel and the fading in a local area. The simple relationship is exact for small-scale fading, regardless of the complexity of the angular power distribution. Eqn (11) has many implications for analysis and measurement in wireless communications. For example, propagation measurements that obtain angle-of-arrival information can yield immediate insight into local area fading characteristics. Conversely, by studying the fading rate of power measurements made along orthogonal tracks, it is possible to infer basic angle-of-arrival channel characteristics.

## Appendix

Eqn (9) may be rewritten as the following:

$$\sigma^2 = 2k^2v^2 \sum_{i=1}^N \sum_{j=1}^N P_i P_j \int_0^{2\pi} \int_0^{2\pi} \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) \delta(\theta_1 - \theta_i) \delta(\theta_2 - \theta_j) d\theta_1 d\theta_2 \quad (12)$$

which may be regrouped, using Eqn (10) and the identity  $\sin^2 \frac{x}{2} = \frac{1}{2} - \frac{1}{2} \cos x$ :

$$\sigma^2 = k^2v^2 \left[ \int_0^{2\pi} \int_0^{2\pi} p(\theta_1) p(\theta_2) d\theta_1 d\theta_2 - \int_0^{2\pi} \int_0^{2\pi} p(\theta_1) p(\theta_2) \cos(\theta_1 - \theta_2) d\theta_1 d\theta_2 \right] \quad (13)$$

Using the identity  $\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$  and rearranging terms leads to

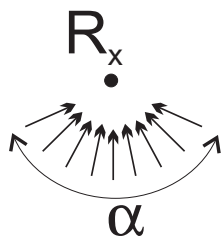
$$\sigma^2 = k^2v^2 \left\{ |F_0|^2 - |F_1|^2 \right\} \quad (14)$$

where  $F_0$  and  $F_1$  are the zeroeth and first complex Fourier coefficients of the angular distribution of power,  $p(\theta)$ . Note that the average local area power,  $P_T$ , is equal to  $|F_0|$ . Eqn (11) is the final result in terms of angular spread,  $\Lambda$ .

## References

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Spatial Representation  
of Arriving Power



Angular Distribution  
of Power

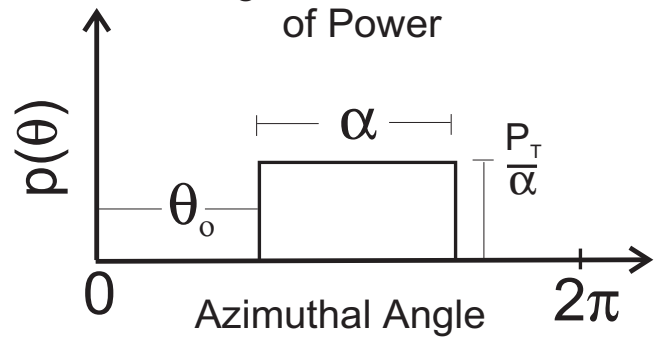


Figure 1: Angular distribution of power,  $p(\theta)$ , for a continuous distribution of multipath components.

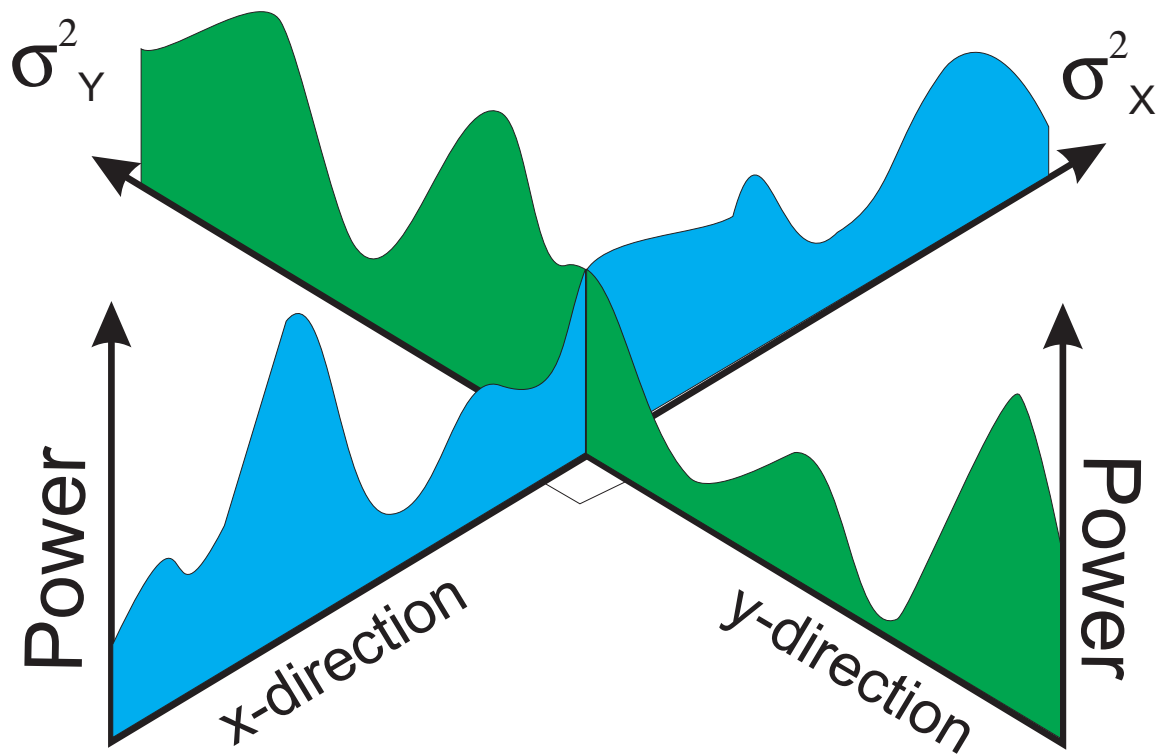


Figure 2: An estimate of  $\sigma^2$  may be obtained by averaging the fading rate variance measured along two orthogonal directions as functions of space or time (with identical velocities); under these conditions,  $\sigma^2 = \frac{1}{2}(\sigma_x^2 + \sigma_y^2)$  regardless of the azimuthal orientation of the measurement.